

1 a.

4ε	_____	n_4
3ε	_____	n_3
2ε	_____	n_2
ε	_____	n_1
0	_____	n_0

Distinguishable particles

Number particles $N = 5$

Total energy $U = 4ε$.

Distributions		0	ε	2ε	3ε	4ε	No microstates.	(2)
(i)	1.	4	-	-	-	1	(ii) 5	5
	2.	3	1	-	1	-	20	20
	3.	3	-	2	-	-	$5C_2 = \frac{5 \times 4}{1 \times 2} = 10$	10
(2) distinguish	4.	2	2	1	-	-	$5 \times 4C_2 = 5 \times \frac{4 \times 3}{1 \times 2} = 30$	30
	5.	1	4	-	-	-	5	5
							<u>70</u>	<u>70</u>

Total number microstates = 70.

(iii) Evaluate Population:

$$n_0 = \frac{5}{70} \times 4 + \frac{20}{70} \times 3 + \frac{10}{70} \times 3 + \frac{30}{70} \times 2 + \frac{5}{70} \times 1 = \frac{175}{70} = 2.50$$

$$n_1 = \frac{20}{70} \times 1 + \frac{30}{70} \times 2 + \frac{5}{70} \times 4 = \frac{100}{70} = 1.43$$

$$n_2 = \frac{10}{70} \times 2 + \frac{30}{70} \times 1 = \frac{50}{70} = 0.72$$

$$n_3 = \frac{20}{70} = 0.29$$

$$n_4 = \frac{5}{70} = 0.07$$

(iv) For bosons
1 microstate
per distribⁿ

See next
sheet for
populations.

Populations
(distinguishable) (2)

1(a) contd.

(iv) Bosons - 1 microstate/distribution

5 microstates in total

$$n_0 = \frac{1}{5} \{4 + 3 + 3 + 2 + 1\} = 13/5 = 2.6$$

$$n_1 = \frac{1}{5} \{1 + 2 + 4\} = 1.40$$

$$n_2 = \frac{1}{5} \{1 + 2\} = 0.6$$

$$n_3 = \frac{1}{5} = 0.2$$

$$n_4 = 1/5 = 0.2$$

Population

(indistinguishable)

(2)

1 (b)

$$2e \text{ ————— } 2n_2$$

$$e \text{ ————— } n_1$$

$$0 \text{ ————— } n_0$$

$$(i) \quad n_1 = n_0 \exp(-\epsilon/kT)$$

$$0.1 = \exp\left(-\frac{1.38 \times 10^{-21}}{1.38 \times 10^{-23} T}\right)$$

$$\ln(0.1) = -2.30 = -\frac{100}{T}$$

$$T = \frac{100}{2.3} = 43.5 \text{ K} \quad (2)$$

$$(ii) \quad n_2 = n_0 \exp(-2\epsilon/kT)$$

$$10^{-6} = \exp\left(-\frac{2 \times 1.38 \times 10^{-21}}{1.38 \times 10^{-23} T}\right)$$

$$\ln(10^{-6}) = -13.82 = -\frac{200}{T}$$

$$T = +\frac{200}{13.82} = 14.5 \text{ K} \quad (2)$$

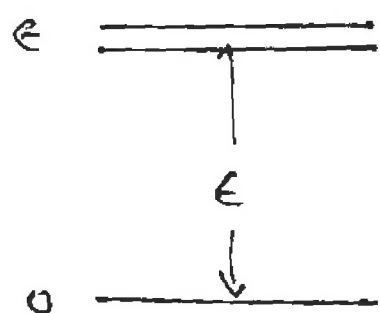
$$(iii) \quad n_2 = n_1 \exp(-\epsilon/kT)$$

$$0.99 = \exp\left(-\frac{1.38 \times 10^{-21}}{1.38 \times 10^{-23} T}\right)$$

$$\ln(0.99) = -0.0100 = -\frac{100}{T}$$

$$T = \frac{100}{0.01} = 10^4 \text{ K} \quad (2)$$

(c)

(i) Partition function Z

$$Z = 1 + 2 \exp(-\epsilon/kT)$$

(2)

$$(ii) U = NkT^2 \frac{\partial}{\partial T} (\ln Z) = NkT^2 \frac{\partial}{\partial T} \ln [1 + 2 \exp(-\epsilon/kT)]$$

$$U = NkT^2 \left\{ \frac{2 \cdot (\epsilon/kT^2) \exp(-\epsilon/kT)}{[1 + 2 \exp(-\epsilon/kT)]} \right\}$$

$$= \frac{2N\epsilon \exp(-\epsilon/kT)}{[1 + 2 \exp(-\epsilon/kT)]}$$

(2)

(iii) Limits

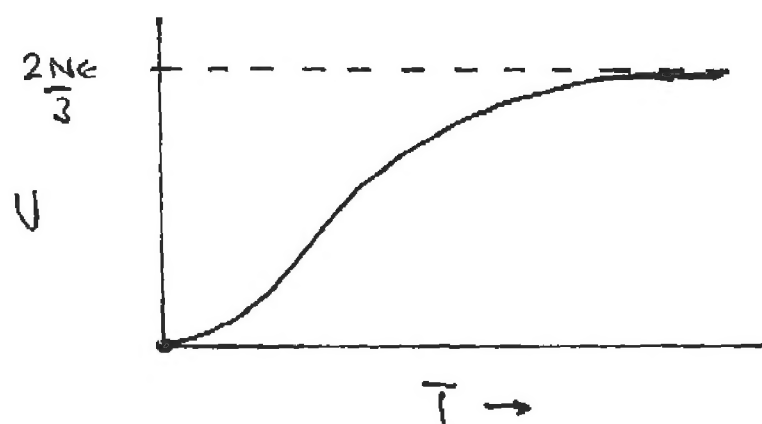
$$\text{As } T \rightarrow 0 \quad \exp(-\epsilon/kT) \rightarrow 0 \quad U \rightarrow 0$$

(1)

$$\text{As } T \rightarrow \infty \quad \exp(-\epsilon/kT) \rightarrow 1 \quad U \rightarrow \frac{2N\epsilon}{3}$$

(1)

(iv)



(2)

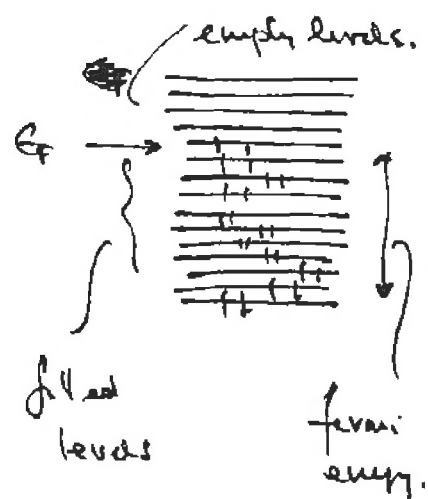
$$C_V = \left(\frac{\partial U}{\partial T} \right)$$

(v)

 C_V is slope of U vs T C_V 

(2)

(d) Fermi energy of a system of electrons.



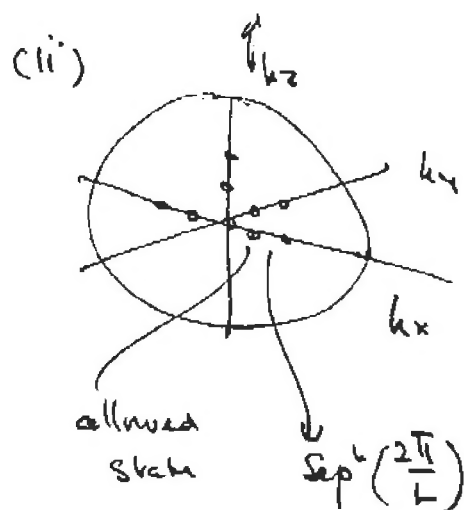
When N electrons populate set of quantised states — they obey Pauli principle.

Only \uparrow and \downarrow electrons in each state.

Filling these states with N electrons — fills them up to certain level of energy E_F .

This E_F is fermi energy.

(i) 2 arises because of ~~total~~ population of k states with \uparrow and \downarrow electrons.



Number of k states up to k_F

$$= \frac{4\pi k_F^3}{3} \cdot \frac{1}{\left(\frac{2\pi}{L}\right)^3}$$

Population of electron $N = 2 \times \frac{4\pi k_F^3}{3} \cdot \frac{V}{(2\pi)^3}$

$$k_F^3 = \frac{3N\pi^2}{V} ; k_F = \left(\frac{3\pi^2 N}{V}\right)^{1/3}$$

(ii) $E_F = \frac{\hbar^2 k_F^2}{2m}$

(iv) Na. 23 kg Na contains 6×10^{26} Na atoms

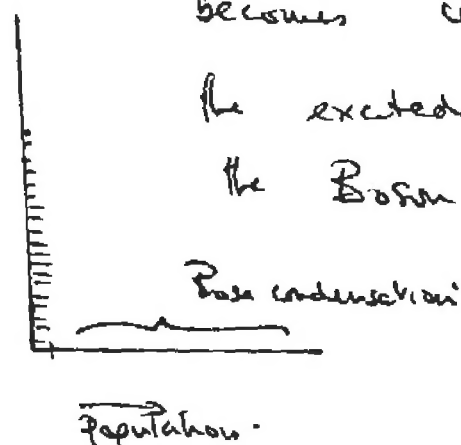
1m³ \approx 971 kg Na contains $\frac{6.02 \times 10^{26}}{23} \times 971 = 2.54 \times 10^{28}$ atoms.

$$\frac{N}{V} = 2.54 \times 10^{28}$$

$$E_F = \frac{\hbar^2}{2m} \left(\frac{3\pi^2 \times 2.54 \times 10^{28}}{1} \right)^{2/3} = \left(\frac{6.63 \times 10^{-34}}{2\pi} \right)^2 \cdot \frac{1}{2 \times 9.11 \times 10^{-31}} \left(\frac{3\pi^2 \times 2.54 \times 10^{28}}{1} \right)^{2/3}$$

1(a). Bose condensation.

Below a critical temperature T_B in a boson system - the population of the ground state of a series of quantised states becomes very large and discontinuous with the population of the excited states. This extra population (non-continuous) is the Bose condensation.



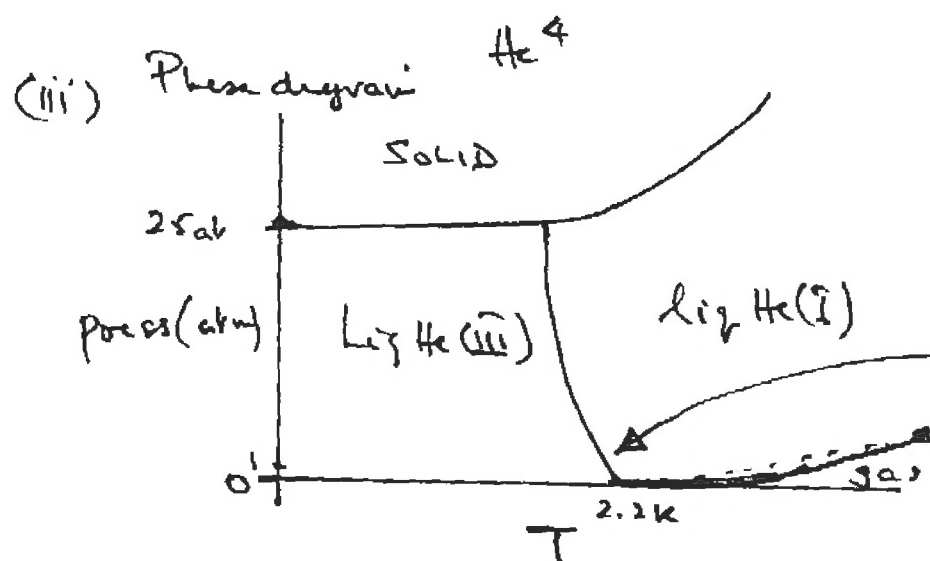
$$(ii) \quad T_B = \left(\frac{h^2}{2\pi m k} \right) \left(\frac{N}{2.612 V} \right)^{2/3}$$

For He^4 liquid $27 \times 10^{-6} m^3$ contains 6×10^{23} atoms.

$$\text{In } 1 m^3 \text{ there are } \frac{6.02 \times 10^{23}}{27 \times 10^{-6}} = 2.23 \times 10^{28} = \frac{N}{V}$$

$$T_B = \left(\frac{(6.63 \times 10^{-34})^2}{2\pi \times 4 \times 1.67 \times 10^{-27} \times 1.38 \times 10^{-23}} \right) \left(\frac{2.23 \times 10^{28}}{2.612} \right)^{2/3}$$

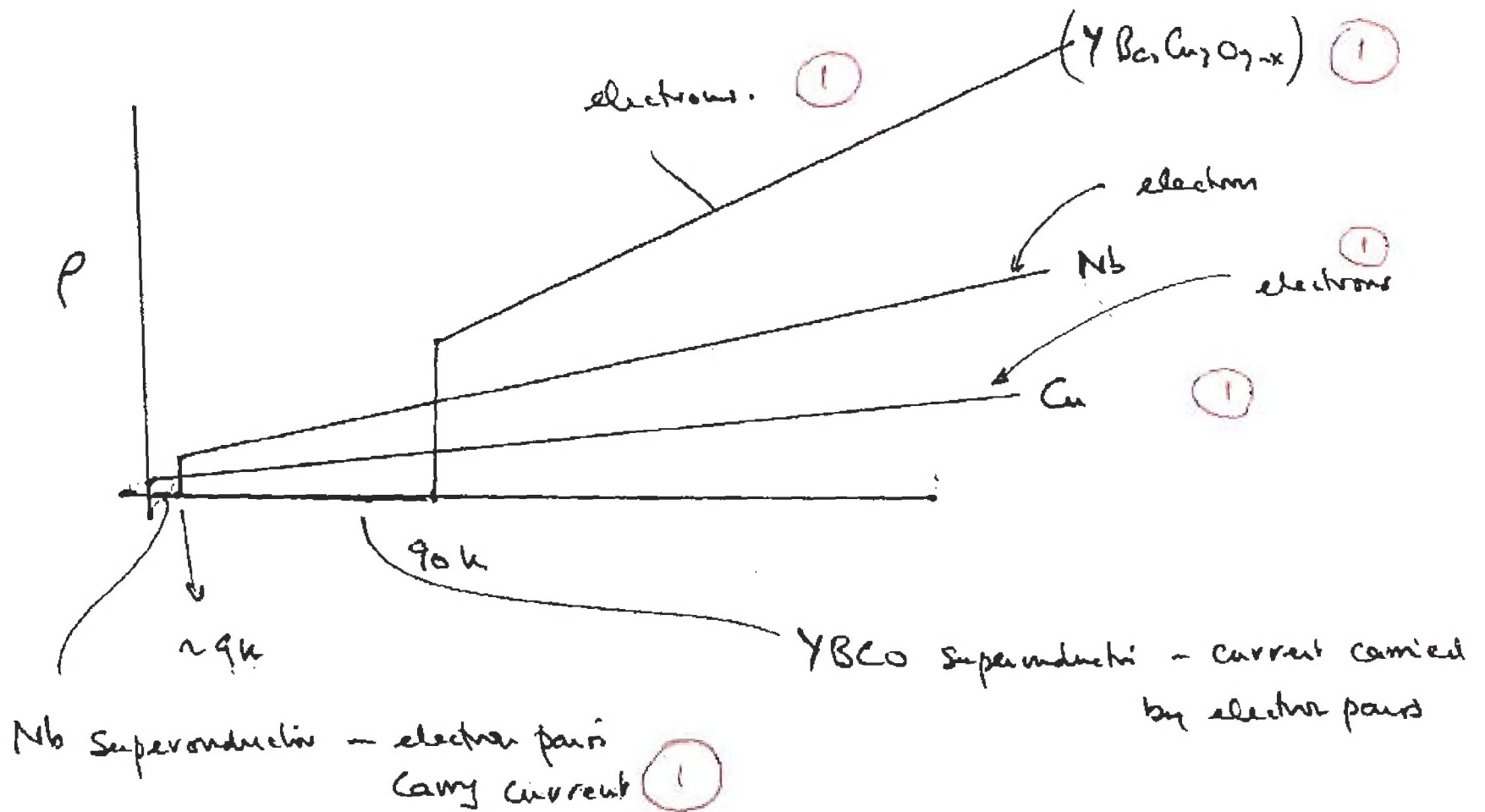
$$T_B = 3.2 K$$



(iv) Bose condensation associated with phase change $Liq He(I)$ to $Liq He(II)$ - the λ line.

1(f). Current carriers.

- (i) In Superconductor current carriers are electron pairs. (1)
- (ii) In normal conductor current carriers are single electrons (1)



$$2(a) \quad k = \frac{2\pi}{\lambda} \quad \lambda = \frac{h}{mv} \quad k = \frac{2\pi mv}{h}$$

$$V = \left(\frac{h}{2\pi} \right) \frac{k}{m} = \frac{\hbar k}{m} \quad (1)$$

$$E = \frac{1}{2} mv^2 \quad (1)$$

$$E = \frac{1}{2} m \cdot \frac{\hbar^2 k^2}{m^2} = \frac{\hbar^2 k^2}{2m} \quad (1)$$

$$(ii) \quad g(k) dk = \frac{4\pi V k^2 dk}{(2\pi)^3}$$

$$k = \frac{mv}{\hbar} \quad dk = \left(\frac{m}{\hbar} \right) dv \quad (1)$$

$$g(v) dv = V \cdot \frac{4\pi}{(2\pi)^3} \cdot \frac{m^2 v^2}{\hbar^2} \cdot \frac{m}{\hbar} \cdot dv \quad (1)$$

$$g(v) dv = 4\pi V \left(\frac{m}{h} \right)^3 v^2 dv \quad (1)$$

$$(iii) \quad Z = \int_0^\infty g(k) dk \exp\left(-\frac{\hbar^2 k^2}{2mkT}\right)$$

$$= \int_0^\infty \frac{4\pi V k^2 dk}{(2\pi)^3} \cdot \exp\left(-\frac{\hbar^2 k^2}{2mkT}\right) \quad (1) \quad I_2 \text{ with } b = \frac{\hbar^2}{2mkT}.$$

$$Z = \frac{4\pi}{(2\pi)^3} \cdot V \cdot \frac{1}{2 \left(\frac{\hbar^2}{2mkT} \right)} \cdot \frac{1}{2} \left(\frac{\pi}{\frac{\hbar^2}{2mkT}} \right)^{1/2} \quad (1)$$

$$= \frac{4\pi V}{(2\pi)^3} \cdot \frac{(2mkT)}{2\hbar^2} \cdot \frac{1}{2} \cdot \left(\frac{2mkT \cdot \pi}{\hbar^2} \right)^{1/2}$$

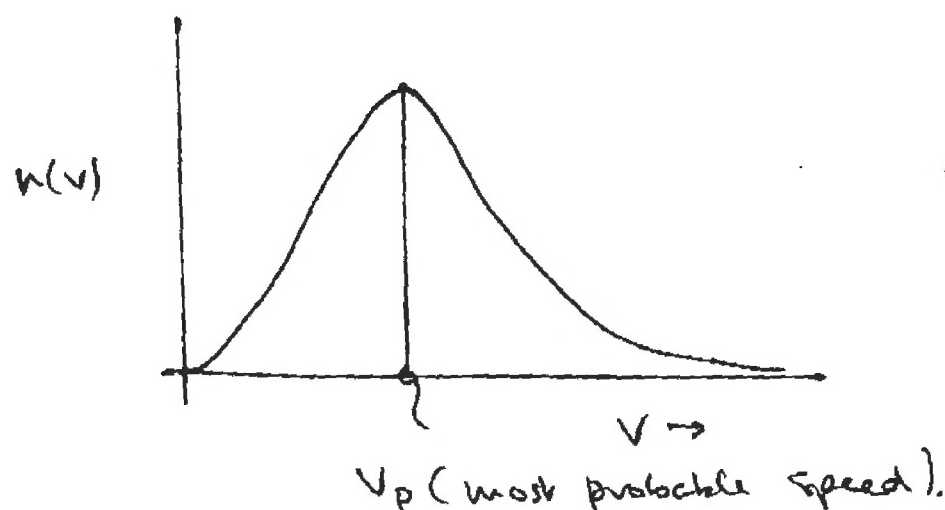
$$Z = \frac{4\pi V}{\cancel{4}} \cdot \frac{(2mkT)}{\hbar^2} \cdot \left(\frac{2\pi mkT}{h^2} \right)^{1/2} = V \cdot \left(\frac{2\pi mkT}{h^2} \right)^{3/2} \quad (1)$$

(iv) Distribution of speeds.

$$n(v) dv = 4\pi \cdot V \cdot \left(\frac{m}{h}\right)^3 v^2 dv \cdot \frac{N}{Z} \exp\left(-\frac{mv^2}{2kT}\right). \quad (1)$$

$$= 4\pi \cancel{V} \cdot \frac{N}{\cancel{V}} \left(\frac{m}{h}\right)^3 \cdot v^2 \exp\left(-\frac{mv^2}{2kT}\right) dv \cdot \frac{\cancel{h^3}}{(2\pi mkT)^{3/2}} \quad (1)$$

$$= 4\pi N \cdot \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 \exp\left(-\frac{mv^2}{2kT}\right) dv \quad (1)$$



(vi) Mean square speed.

$$b = + \frac{m}{2kT}$$

$$\langle v^2 \rangle = \frac{\int_0^\infty v^2 n(v) dv}{\int_0^\infty n(v) dv} = \frac{\text{mols}}{\text{Counts}} \frac{\int_0^\infty v^4 \exp\left(-\frac{mv^2}{2kT}\right) dv}{\int_0^\infty v^2 \exp\left(-\frac{mv^2}{2kT}\right) dv} \quad (1)$$

$$\langle v^2 \rangle = \frac{I_1}{I_2} = \frac{(n-1)}{2b} \stackrel{n=4}{=} \frac{3}{2 \left(\frac{m}{2kT}\right)} = \frac{3kT}{m}. \quad (1)$$

$$\langle v^2 \rangle_{\text{He at } 300\text{K}} = \frac{3 \times 1.38 \times 10^{-23} \times 300}{4 \times 1.66 \times 10^{-27}} = 1.87 \times 10^6 \text{ m}^2/\text{s}^2 \quad (1)$$

(vii) Energy / mole $U = N \cdot \frac{1}{2} m \langle v^2 \rangle$

$$= \frac{N}{2} \cdot \frac{3kT}{1} = \frac{3}{2} NkT. \quad (1)$$

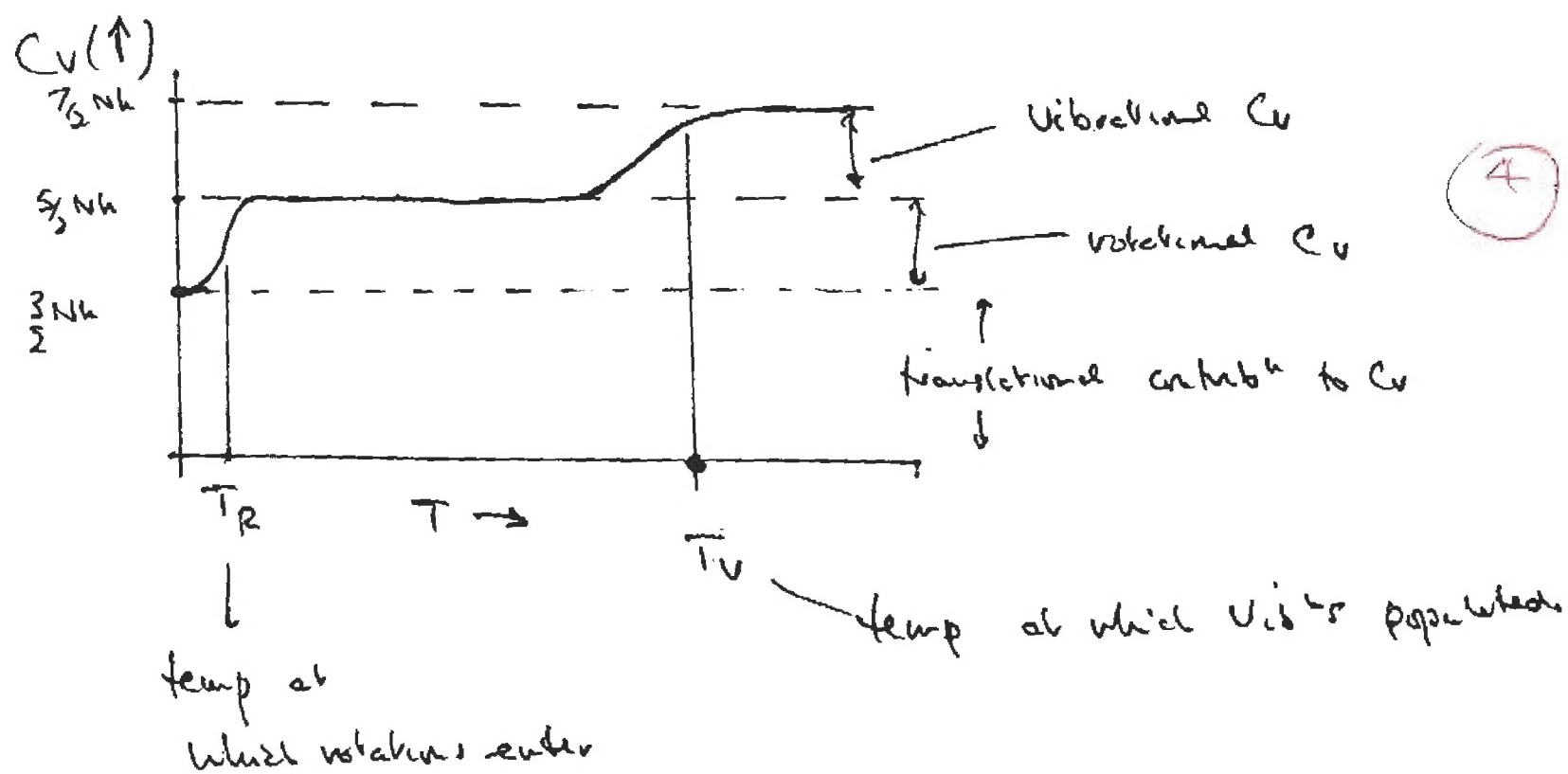
$$C_v / \text{mole} = \left(\frac{\partial U}{\partial T} \right) = \frac{3}{2} Nk. \quad (1)$$

(viii) Additional excitations in diatomic gas.

Vibration $\leftarrow \text{O} \quad \text{O} \rightarrow$ (1)

Rotation $\curvearrowright \quad \curvearrowleft$ (1)

(ix)



Thus as $T \rightarrow \infty$

$$U \rightarrow \frac{12\pi^5 V k^4 T^4}{15 c^3 h^3} \cdot \frac{1}{3} \cdot \frac{N}{V} \cdot \frac{1}{4\pi}$$

$$U \rightarrow 3NkT.$$

(1)

(vii) $C_v = \left(\frac{\partial U}{\partial T} \right)_v$

As $T \rightarrow 0$ $C_v \rightarrow \frac{\partial}{\partial T} \left(\frac{16\pi^5 V k^4 T^4}{15 c^3 h^3} \right)$

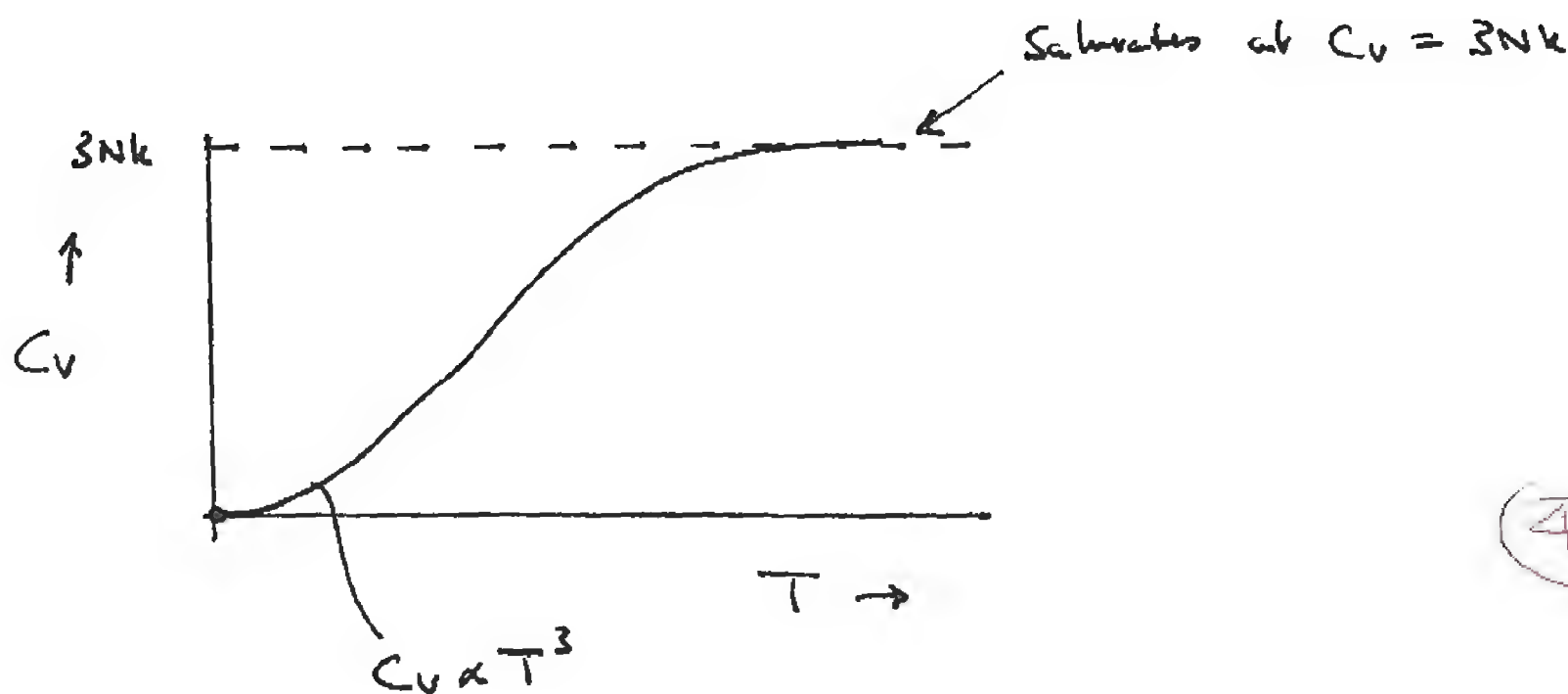
$$C_v \rightarrow \frac{16\pi^5 V}{15} \left(\frac{k}{hc} \right)^3 k T^3.$$

(1)

As $T \rightarrow \infty$ $C_v \rightarrow 3Nk.$

(1)

(ix)



(4)

As $T \rightarrow 0$ $C_v \rightarrow 0$ as $C_v \propto T^3$ — good agreement with data
 As $T \rightarrow \infty$ $C_v \rightarrow 3Nk$ — agrees with expt.

Between — shape quite good but not exact.

Overall — not simple theory but $C_v \propto T^3$

2(b) Conditions for quantised phonons

$$(i) \quad k_x = \frac{2\pi n_x}{L}, \quad k_y = \frac{2\pi n_y}{L}, \quad k_z = \frac{2\pi n_z}{L} \quad (1)$$

where $n_x, n_y, n_z = \pm 1, \pm 2, \pm 3, \dots$

$$(ii) \quad g(k) dk = \frac{\text{volume of } k \text{ space}}{\text{space/allowed state}} = \frac{4\pi k^2 dk}{\left(\frac{2\pi}{L}\right)^3} \quad (1)$$

$$g(k) dk = \frac{4\pi \cdot V \cdot k^2 dk}{(2\pi)^3} \quad (1)$$

Take account of longitudinal \leftrightarrow and transverse \nleftrightarrow modes.

$$g(k) = 3 \times \frac{4\pi V k^2 dk}{(2\pi)^3} \quad (1)$$

$$(iii) \quad v = \frac{kc}{2\pi} \quad k = \frac{2\pi v}{c} \quad dk = \frac{2\pi}{c} dv \quad (1)$$

$$g(v) dv = 3 \cdot \frac{4\pi V}{(2\pi)^3} \left(\frac{2\pi v}{c}\right)^2 \left(\frac{2\pi}{c}\right) dv \quad (1)$$

$$= \frac{12\pi V v^2 dv}{c^3} \quad (1)$$

For cut off frequency

$$3N = \int_0^{v_D} g(v) dv = \int_0^{v_D} \frac{12\pi V}{c^3} v^2 dv = \frac{12\pi V}{c^3} \frac{v_D^3}{3} \quad (1)$$

$$v_D^3 = \frac{3N c^3}{4\pi V} \quad (1)$$

(iv) $h\nu$ = quantum of photon energy

(1)

$\frac{1}{[\exp(h\nu/kT) - 1]}$ = boson probability of occupation of state of energy $h\nu$ at temperature T .

(1)

(v) Internal energy $U = \int_0^{\nu_D} \frac{12\pi V}{c^3} \cdot \nu^2 d\nu \cdot h\nu \cdot \frac{1}{[\exp(h\nu/kT) - 1]}$

(1)

Put $y = h\nu/kT$ $y_D = \frac{h\nu_D}{kT}$

$dy = \frac{h}{kT} d\nu$ or $d\nu = \frac{kT}{h} dy$

(1)

Thus $U = \frac{12\pi V}{c^3} h \int_0^{y_D} \frac{\left(\frac{kT}{h}\right)^3 y^3 \left(\frac{kT}{h}\right) dy}{[\exp(y) - 1]}$

$U = \frac{12\pi V}{c^3} \frac{k^4 T^4}{h^3} \int_0^{y_D} \frac{y^3 dy}{[\exp(y) - 1]}$

(1)

(vi) As $T \rightarrow 0$ $y \rightarrow \infty$

Thus $\int_0^{y_D} \frac{y^3 dy}{[\exp(y) - 1]} \rightarrow \int_0^{\infty} \frac{y^3 dy}{[\exp(y) - 1]} = \frac{\pi^4}{15}$

(1)

Thus $U \rightarrow \frac{12\pi V}{c^3} \frac{k^4 T^4}{h^3} \cdot \frac{\pi^4}{15} = \frac{4\pi^5 V}{15} \frac{k^4 T^4}{c^3 h^3}$

(1)

(vii) As $kT \gg h\nu$ $y \rightarrow 0$ $\int_0^{y_D} \frac{y^3 dy}{[\exp(y) - 1]} \rightarrow \int_0^{y_D} \frac{y^3 dy}{[1 + y + \dots - 1]} = \frac{y_D^4}{4}$

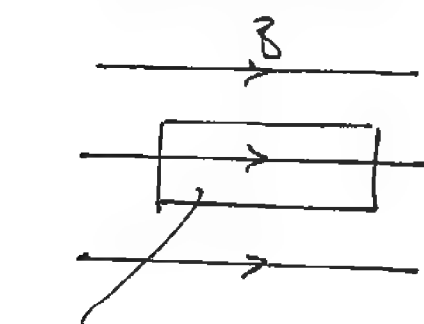
(1)

Then $U \rightarrow \frac{12\pi V}{c^3 h^3} \frac{k^4 T^4}{2} \left(\frac{h^3 \nu_D^3}{15 k T^3} \right)$ but $\nu_D^3 = \frac{3c^3 \cdot N}{15 k T^3}$

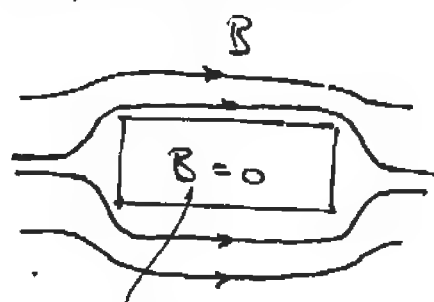
3(a)

(i)

Meissner effect in superconductors.



Sample is normal state



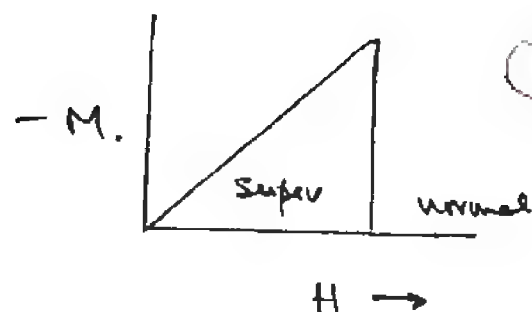
Sample is Superconducting State

When sample is cooled below critical temperature T_c is

Superconducting it expels all lines of B.

In terms of M, H $B = \mu_0(H + M)$

In superconductor $B = 0$ $M = -H$.



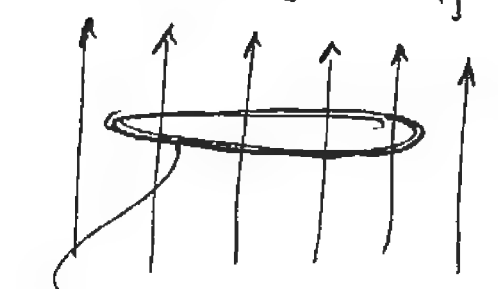
Graph of $-M$ vs H for superconducting sample.

$-M$ achieved by



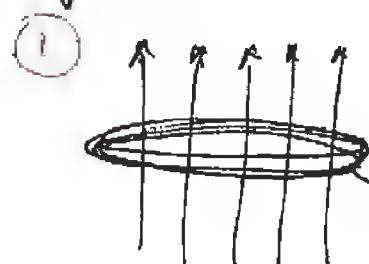
Superconducting currents flowing around surface.

(ii) Magnetic flux quantum



ring sample in superconducting state

field B



Field removed

Current flows in superconductor to conserve field that was present before.

Found that the superconducting current is quantised so that

the number of flux lines of $B \equiv \text{Area of ring} \times B = n \phi_0$

where ϕ_0 is quantum of flux. and n is integer

Measurements determine $\phi_0 = \frac{h}{2e}$

BCS theory predicts $\phi_0 = \frac{h}{2e}$ unit of current carrier.

Consistent that superconducting

(iii) Superconductivity depends on condensation of electron pairs (Cooper pairs) into boson ground state. (1)

However critical temperature is not set by Bose Condensation temperature (which is relatively high) but by interaction that causes electron pairing. (1)

The critical temperature set by $kT_c \approx$ pairing energy. (1)

Pairing interaction caused by electron-lattice-electron interaction which lowers energy of 1 to $k=0$ pairs. (1)

(5) of these

(iv) Experiments supporting the 2 fluid model of liquid He (II)

2 fluid model of liquid He (II) - interpenetrating (1)

Superfluid and normal fluid -

Sets in at $T < T_\lambda$ (1)

At $T = T_\lambda$ all normal - at $T \rightarrow 0$ all superfluid. (1)

Expts - measurement of viscosity (1) by timing oscillation of discs immersed in liq He (II)

Normal fluid adheres to discs, increases moment of inertia & affects the oscillation time. - gives η_{discs} . (1)

- also measure viscosity by flow through small capillary (1)
- Set $\eta_{capillary}$

Found $\eta_{capillary} \ll \eta_{discs}$ because $\eta_{capillary}$ sees superfluid (1)

(5) of these

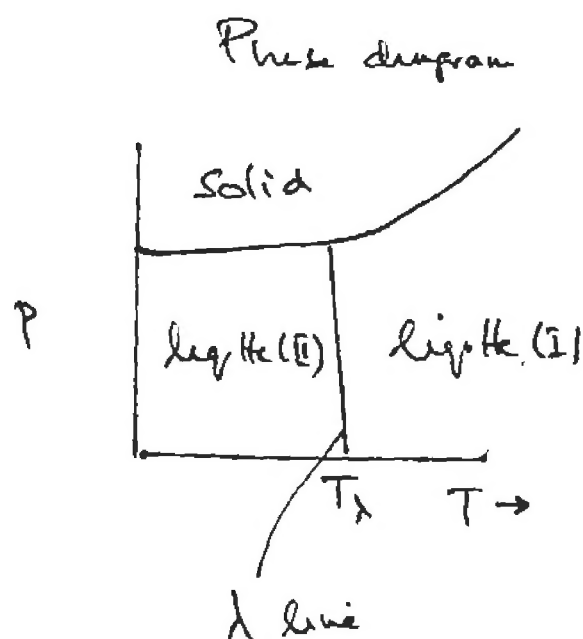
M.D. P. ...

(V) The λ transition in liquid He^4 .

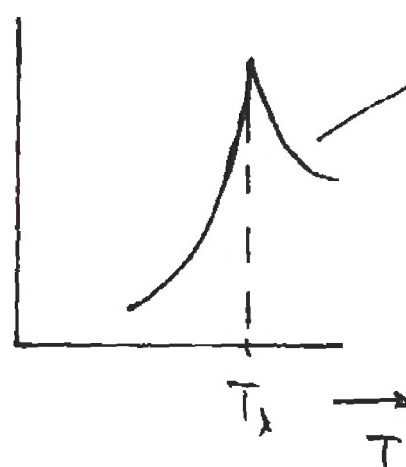
Transition between liq He (I) and liq He (II)

Occurs $\sim T_\lambda \sim 2.2 \text{ K}$.

Internal energy U , and C_v much reduced in liquid He (II) because of superfluid component in liquid He (II) and not in liquid He (I)



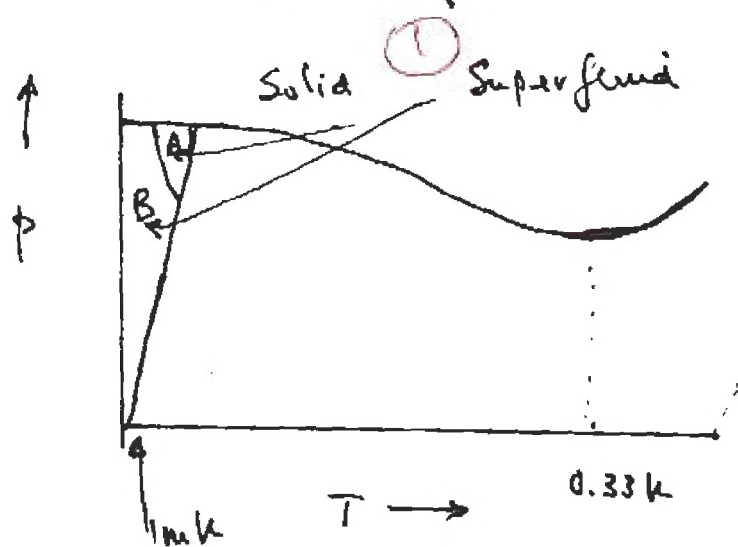
C_v



called λ transition from shape of C_v versus T graph.

(vi) Superfluid He^3 — occurs $T < 1 \text{ mK}$.

Phase diagram



occurs via pairs of He^3 atoms.

which form bosons and undergo Bose condensation.

Pairs have $S=1$ $L=1$.

Complex behaviour — more than one phase A, B.

Phases sensitive to magnetic field

(5) of these

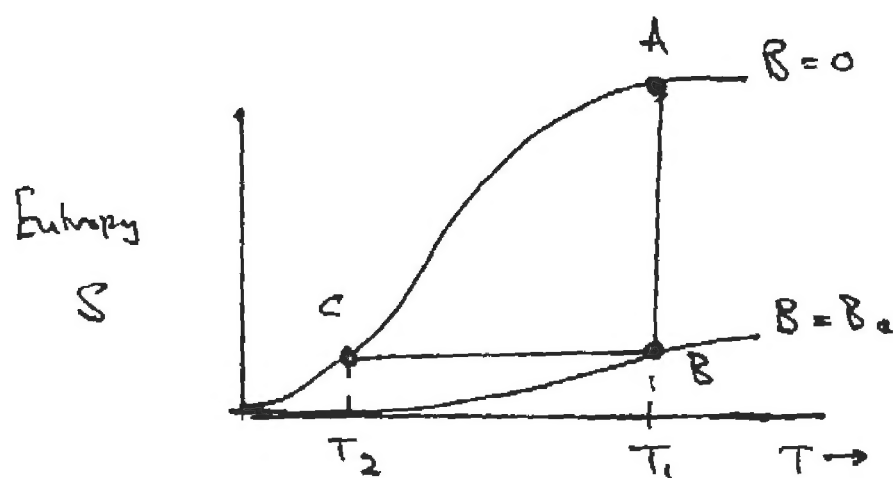
Adiabatic demagnetization.

Some students think it will have negative magnet.

(vi) Sample - can be nuclear or atomic

Choose nuclear Cu metal Cu^{63} nucleus $I = 3/2$.

(vii) Process.



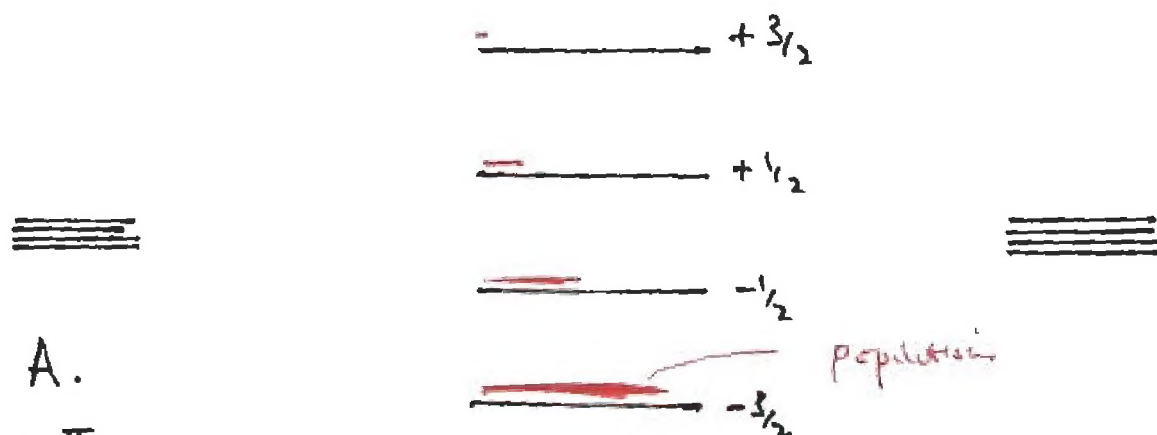
1. Magnetize isothermally at temp T_1
(apply field with sample connected to heat bath at T_1 . $A \rightarrow B$)

2. Isolate sample.

3. Demagnetize adiabatically $B \rightarrow C$
Sample temp drops to T_2 .

Process

(viii) Theory



Theory

1 diagram

A.
Temp T_1
Substates nearly degenerate
Populations nearly same

B
field B_0 applied at T_1
States split.
Populations heavily favored lower energy states

C
Adiabatic (S const)
So populations stay same.
These populations with small splitting \rightarrow low temp T_2

(ix) Populations in B + C determined by B/T .

But populations same. - Thus $(B/T)_B = (B/T)_C$

$$\text{Hence } \frac{10.0}{2 \times 10^{-3}} = \frac{0.5 \times 10^{-3}}{T_2}$$

$$T_2 = \frac{10 \times 10^{-6}}{10} = 1 \times 10^{-6} \text{ K.}$$

Calc

1

1

12

OK (3a)

Helium dilution refrigerator

- (i) Form of helium - mixture of He^3 and He^4 liquids ^①
+ ^①
(ii) Theory of cooling

Below triple point $T = 0.96 \text{ K}$ He^3 and He^4 liquids form
a 2 phase system ^① — He^3 rich phase
 He^4 rich phase

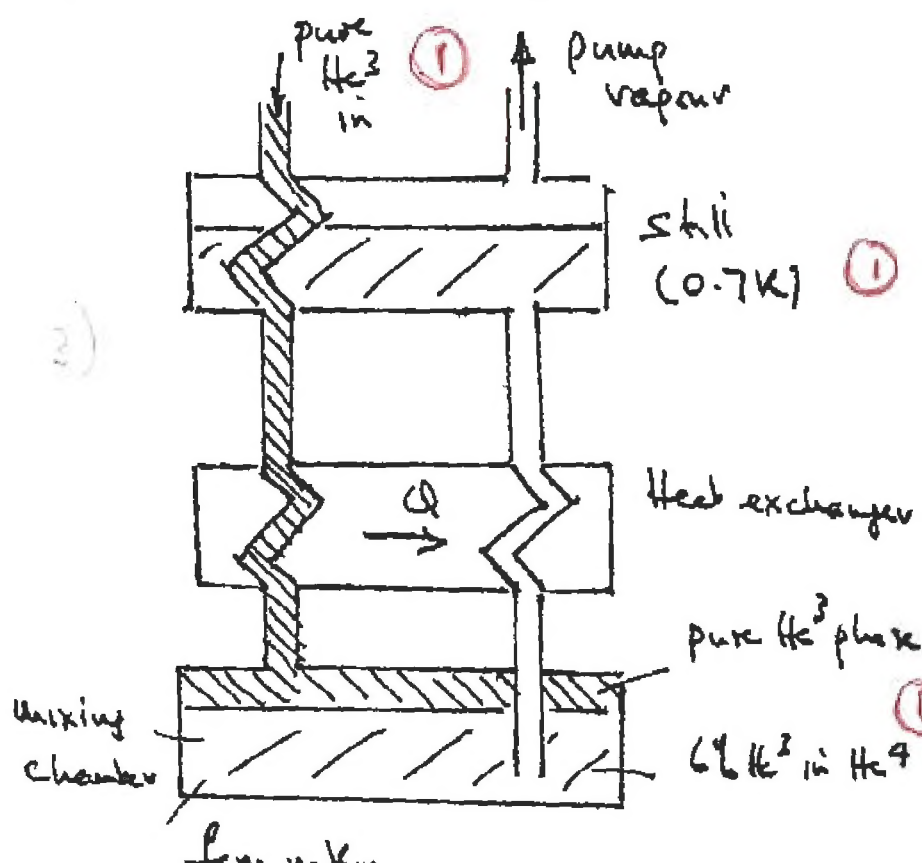
At $T = 0$ He^3 rich phase $\rightarrow 100\% \text{ He}^3$ ^①
 He^4 rich phase $\rightarrow 6\% \text{ He}^3 + 94\% \text{ He}^4$ ^①

Cooling.

In mixing chamber (see diagram) phase of He^3 floats on top of
 He^4 ($6\% \text{ He}^3$) phase ^①

Cooling occurs at phase boundary as He^3 atoms evaporate
from He^3 rich to He^4 rich phases. ^① This takes energy
from He^3 rich phase and cools it. Theory any ^④ of this ^⑥

(iii) Schematic diagram



(iv) Experimental process.

Apparatus works by allowing
above evaporation to occur
continuously. (in cycle)

Dilute phase pumped —

① pure He^3 gas comes off
and is circulated down

then heat exchanger

① to pure He^3 layer in
mixing chamber — continues
process.

(v) Starting temp $\sim 0.5 \text{ K}$ ^①

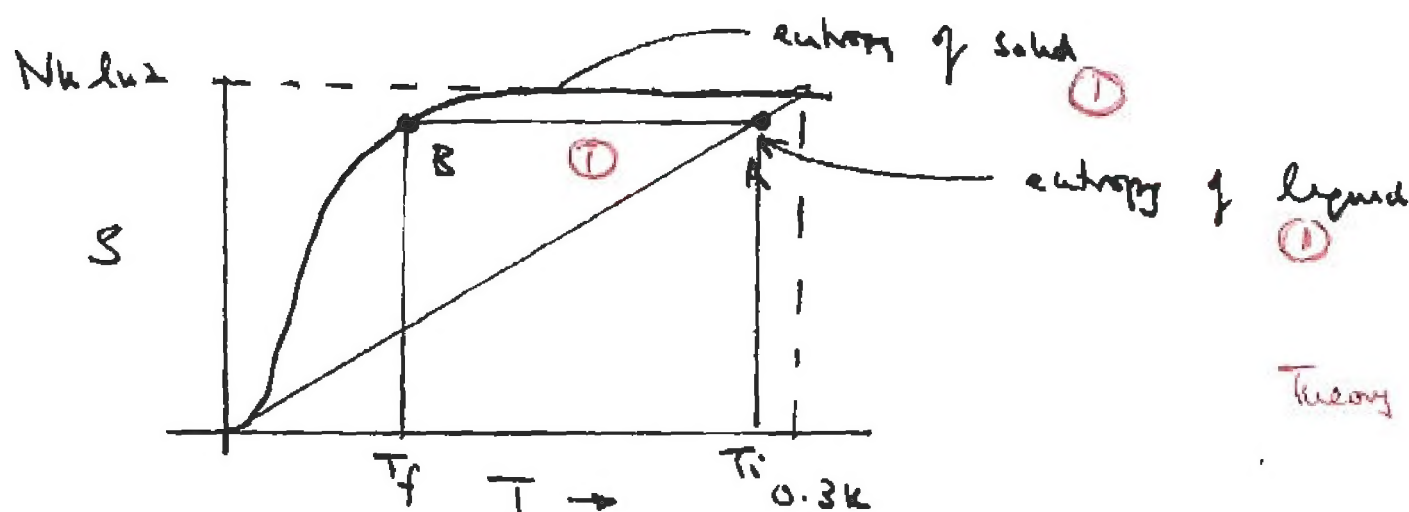
EITHER S(a)

Pomeranchuk cooling.

Uses liquid / solid He^3 . ① + ①

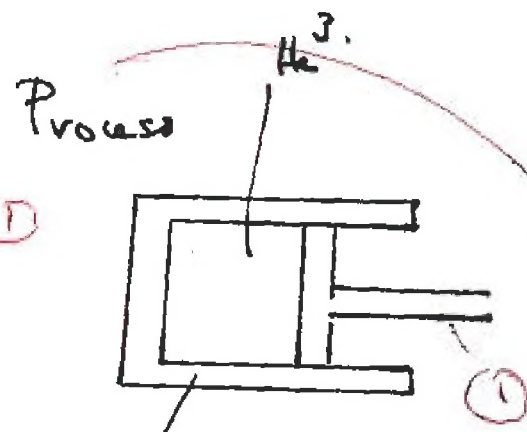
Theory. In temp range $0 \rightarrow 0.3K$ entropy S of solid He^3 is greater than liquid He^3 . ①

This because main entropy is in nuclear spins ① — this greater in Boltzmann distribⁿ in distinguishable atoms than in Fermi-Dirac distribⁿ in indistinguishable liq atoms. ①



Theory any ④ of this ⑥

Physics + theory
solid of ④



① Insulating walls

In practice start ~ ~~0.2K~~ 0.2K and will end at ~ 10 mK.

One shot process. — connect sample to cell via conductivity path ①

Schematic diagram

Type used He^3 ②

Theory + basic procedure ⑥

Schematic diag ③

Steady + final ②

12 marks.

13.